12 A LESSON 3

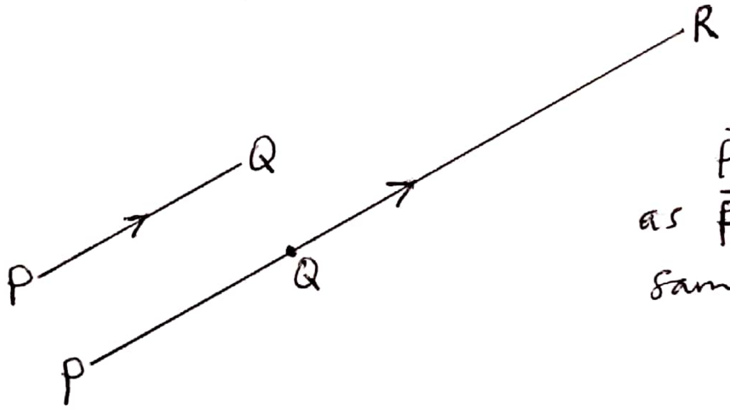
COLLINEARITY AND VECTOR GEOMETRY

When two or more points lie on the same straight line, the points are collinear.

If the points P, Q and R are collinear, then $\overrightarrow{PQ} = K\overrightarrow{QR}$. The vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ have the same gradient and they also share a point at Q.

If two vectors $\underline{p}$ and $\underline{q}$ are parallel, then $\underline{p} = \alpha \underline{q}$, where α is a scalar.

$$\overrightarrow{PR} = 3\overrightarrow{PQ}$$



$\overrightarrow{PQ} : \overrightarrow{QR} = 1 : 2$

$\overrightarrow{PR}$ is 3 times as long as $\overrightarrow{PQ}$ and has the same direction.

Example

Show that the points P(-3,-11), Q(1,1) and R(4,10) are collinear.

Solution:

The position vectors of P(-3,-11), Q(1,1) and R(4,10) are as follows

$\overrightarrow{OP} = \begin{pmatrix} -3 \\ -11 \end{pmatrix}$, $\overrightarrow{OQ} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\overrightarrow{OR} = \begin{pmatrix} 4 \\ 10 \end{pmatrix}$

$$\overrightarrow{PO} = -\overrightarrow{OP} = -\begin{pmatrix}-3\\-11\end{pmatrix} = \begin{pmatrix}3\\11\end{pmatrix}$$

$$\overrightarrow{QO} = -\overrightarrow{OQ} = -\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\end{pmatrix}$$

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$$
$$\overrightarrow{PQ} = \begin{pmatrix}3\\11\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix}$$
$$\overrightarrow{PQ} = \begin{pmatrix}4\\12\end{pmatrix}$$
$$\overrightarrow{PQ} = 4\begin{pmatrix}1\\3\end{pmatrix}$$

$$\overrightarrow{QR} = \overrightarrow{QO} + \overrightarrow{OR}$$
$$\overrightarrow{QR} = \begin{pmatrix}-1\\-1\end{pmatrix} + \begin{pmatrix}4\\10\end{pmatrix}$$
$$\overrightarrow{QR} = \begin{pmatrix}3\\9\end{pmatrix}$$
$$\overrightarrow{QR} = 3\begin{pmatrix}1\\3\end{pmatrix}$$

Since $\overrightarrow{PQ} = 4\begin{pmatrix}1\\3\end{pmatrix}$ and $\overrightarrow{QR} = 3\begin{pmatrix}1\\3\end{pmatrix}$ and Q is common to $\overrightarrow{PQ}$ and $\overrightarrow{QR}$, then P, Q and R must be collinear points.

Example

$\underline{p} = \begin{pmatrix}1\\-4\end{pmatrix}$ and $\underline{r} = \begin{pmatrix}m\\2\end{pmatrix}$. Given that $\underline{p}$ is parallel to $\underline{r}$, find m.

Solution

If $\underline{p}$ is parallel to $\underline{r}$, then $\underline{p} = \alpha\underline{r}$ where α is a scalar.

$$\underline{p} = \alpha \underline{r} \Rightarrow \begin{pmatrix} 1 \\ -4 \end{pmatrix} = \alpha \begin{pmatrix} m \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} \alpha m \\ 2\alpha \end{pmatrix} \qquad \therefore \alpha m = 1 \quad \text{and} \quad 2\alpha = -4$$

first solve $2\alpha = -4$

$$\frac{2\alpha}{2} = \frac{-4}{2}$$

$$\alpha = -2$$

substituting α in $\alpha m = 1$ we have

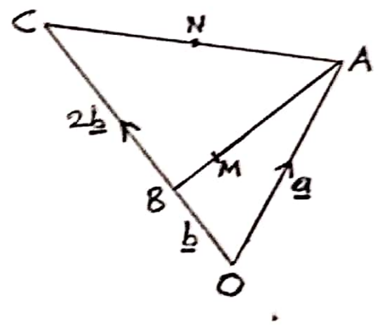
$$-2m = 1$$

$$\frac{-2m}{-2} = \frac{1}{-2}$$

$$m = -\frac{1}{2}$$

Example

In the figure below, OAB is a triangle. OB is produced to point C. CNA is a straight line. Given that $\overrightarrow{OC} = 3\overrightarrow{OB}$ and that $\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OB} = \underline{b}$, AM : MB = 3 : 1, AN = C.



(a) Find in terms of $\underline{a}$ and/or $\underline{b}$

(i) $\overrightarrow{AB}$

(ii) $\overrightarrow{AC}$

(iii) $\overrightarrow{OM}$

(iv) $\overrightarrow{ON}$

(b) Hence show that, O, M and N are collinear

Solution

(a) (i) $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -\underline{a} + \underline{b}$

(ii) $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$

$= -\underline{a} + (b + 2\underline{b})$

$= -\underline{a} + 3\underline{b}$

(iii) $\overrightarrow{OM} = \overrightarrow{OB} + \overrightarrow{BM}$

$= \underline{b} + \frac{1}{4}\overrightarrow{BA}$

$\overrightarrow{BA} = -\overrightarrow{AB} = -(-\underline{a}+\underline{b}) = \underline{a} - \underline{b}$

$\therefore \overrightarrow{BM} = \frac{1}{4}\overrightarrow{BA} = \frac{1}{4}(\underline{a} - \underline{b})$

$= \underline{b} + \frac{1}{4}(\underline{a} - \underline{b})$

$= \underline{b} + \frac{1}{4}\underline{a} - \frac{1}{4}\underline{b}$

$= \frac{1}{4}\underline{a} + \underline{b} - \frac{1}{4}\underline{b}$

$= \frac{1}{4}\underline{a} + \frac{3}{4}\underline{b}$

(iv) $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN}$

$\overrightarrow{AN} = \overrightarrow{NC}$ and $\overrightarrow{AN} + \overrightarrow{NC} = \overrightarrow{AC}$

$\overrightarrow{AN} + \overrightarrow{AN} = \overrightarrow{AC}$

$2\overrightarrow{AN} = \overrightarrow{AC}$

$\overrightarrow{AN} = \frac{1}{2}\overrightarrow{AC}$

$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC}$

$= \underline{a} + \frac{1}{2}(-\underline{a} + 3\underline{b})$

$= \underline{a} - \frac{1}{2}\underline{a} + \frac{3}{2}\underline{b}$

$= \frac{1}{2}\underline{a} + \frac{3}{2}\underline{b}$

(b) $\overrightarrow{OM} = \frac{1}{4}\underline{a} + \frac{3}{4}\underline{b} = \frac{1}{4}(\underline{a} + 3\underline{b})$

$\overrightarrow{ON} = \frac{1}{2}\underline{a} + \frac{3}{2}\underline{b} = \frac{1}{2}(\underline{a} + 3\underline{b})$

$\therefore \overrightarrow{OM} = \frac{1}{2}\overrightarrow{ON}$ and O is common to $\overrightarrow{OM}$ and $\overrightarrow{ON}$

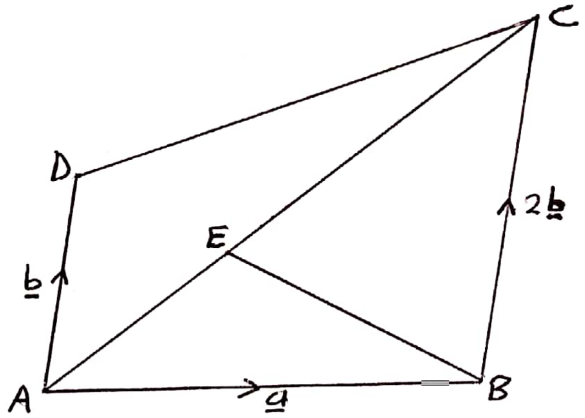
Thus O, M and N are collinear.

Exercise

1. $\underline{a} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and $\underline{c} = \begin{pmatrix} u \\ 10 \end{pmatrix}$. Given that the vector $\underline{c}$ is parallel to the vector $\underline{a}$, calculate the value of u.

2. Show that the points $E(4, 3)$, $F(0, 5)$ and $G(-8, 9)$ are collinear.

3. In the quadrilateral ABCD below, $\overrightarrow{AB} = \underline{a}$, $\overrightarrow{AD} = \underline{b}$, $\overrightarrow{BC} = 2\underline{b}$ and $AE : AC = 1 : 3$



(a) Find in terms of $\underline{a}$ and / or $\underline{b}$

(i) $\overrightarrow{AE}$

(ii) $\overrightarrow{BE}$

(iii) $\overrightarrow{BD}$

(b) Hence or otherwise, show that the points B, E and D are collinear.